vary two rapidly with the time. The period would naturally be less variable than if the radiation emanated only from isolated sunspots, but it might vary considerably if the latitude of most intense radiation varied.

Sunspots may be sources of specially vigorous solar discharges, or they may simply be symptoms that the sun is at the moment or recently has been specially active. Even in the quietest of years, and in the absence of visible sunspots, we have variations in the daily D range, and also at least minor disturbances; and, as the present investigation has conclusively shown, the 27-day period may be not merely recognisable but actually prominent in such a year. In fact 1913, with the lowest sunspot frequency since 1858, had the 27-day period particularly well developed.

It remains to express my indebtedness to the Government Grant Committee for the funds which rendered possible the measurement of the Kew declination ranges from 1858 to 1889. I have also to thank the Astronomer Royal for valuable information about sunspots and for useful references.

On Doubly-Resonated Hot-Wire Microphones.

By E. T. Paris, M.Sc., Fellow of University College, London.

(Communicated by Prof. H. L. Callendar, F.R.S. Received March 30, 1922.)

§ 1. *Introduction.*—In the selective hot-wire microphone* a Helmholtz resonator is used for the detection of sound-waves of definite pitch. The resonator responds to the particular note to which it is tuned, and its response is measured by the change in resistance of an electrically heated platinum-wire grid mounted in the "neck" of the resonator.

The magnitude of the response depends in the usual manner on the tuning and damping of the receiving system, and therefore in order to obtain great sensitivity the damping must be small and consequently the resonance must be sharp. For some purposes, however, sharp resonance may be a disadvantage—as, for example, when the source of sound is liable to small variations in pitch, or when allowance must be made for the Doppler effect—while at the same time it may be desired to retain a high degree of sensitivity. Even in cases where very sharp tuning is permissible, the only effective means of increasing the acoustical sensitivity is by reducing the neck of the resonator, and since this tends to expose the hot-wire grid to chance currents of air, accurate observation becomes difficult except under very favourable conditions. To meet such difficulties as these, it has been found convenient to use doubly-resonated microphones in which the hot-wire grid is placed in a short neck between two resonators, only one of which communicates directly with the outside air. In this way it is possible to increase the range of response, and at the same time to obtain greater sensitivity than is practicable with a singly-resonated microphone.

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* W. S. Tucker and E. T. Paris, 'Phil. Trans.,' A, vol. 221, pp. 389–430 (1921).

This combination of a double resonator with the hot-wire grid, to form a very sensitive microphone, is due to Prof. H. L. Callendar, who carried out experiments early in 1918 to show how the component resonators could be arranged in order to secure a high degree of sensitivity together with a wide range of response.

In Prof. Callendar's experiments the neck of the singly-resonated microphone was inserted into a small hole at the closed end of a resonator of the "stopped pipe" variety. This arrangement was found to be very effective. About the same time, experiments along similar lines were being conducted by Major W. S. Tucker, R.E., who was then in charge of the Acoustical Research Section of the Munitions Inventions Department. In Major Tucker's experiments the neck of a Helmholtz resonator was inserted into a small hole at the narrow end of a conical trumpet. This arrangement also formed a double resonator which differed from that used by Prof. Callendar in that the outer resonator was a cone instead of a parallel pipe.

Since the time when these experiments were performed, the author, working with Major Tucker—first in the Munitions Inventions Department and later at the Signals Experimental Establishment, Woolwich—has had occasion to carry out a number of experiments with double resonators, and to study their properties by means of the hot-wire microphone. It is the purpose of the present paper to give an account of these investigations.

Reference may be made here to a very sensitive instrument constructed by Prof. C. V. Boys* in 1890, consisting of a double resonator fitted with a Rayleigh disc. In this instrument, according to Rayleigh—

> "the sensitiveness is exalted to an extraordinary degree. This is effected . . . . partly by the adoption of double resonance. The large resonator is a heavy brass tube of about 10 cm. diameter, closed at one end, and of such a length as to resound to *e*′. The mirror [Rayleigh's disc] is hung in a short lateral tube forming a communication between the large resonator and a small glass bulb of suitable capacity. The

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* 'Nature,' vol. 42, p. 604 (1890).

> external vibrations may be regarded as magnified first by the large resonator and then again by the small one, so that the mirror is affected by powerful alternating currents of air. The selection of pitch is so definite that there is hardly any response to sounds which are a semi-tone too high or too low."*

Thus the double resonator used by Boys consisted of an outer resonator of the "stopped pipe" variety, communicating near the closed end with a Helmholtz resonator, as in Prof. Callendar's experiments. The properties of this type of double resonator—which for the convenience of reference will be called the "Boys double resonator"—are discussed in §§ 2–6 of the present paper. It may be noted here, in regard to the statement as to the definiteness of the pitch selected by Boys' resonator, that in general all such resonators have *two* natural tones. Under certain conditions, however, when the interval between the tones is large, the response may be much stronger to one tone than to the other. When this is the case, the resonator is often remarkably sensitive to the stronger tone, while the weaker one might even escape detection. The double resonator constructed by Boys appears to have been of this kind.

The present paper deals with the properties of two kinds of double resonator, namely, the Boys resonator (§§ 2–6), and a type which will be called the "Helmholtz double resonator" (§§ 7–10). The latter consists of inner and outer resonators communicating through a narrow tube or neck as in the Boys resonator, but both the components are of the Helmholtz pattern. The theory of this double resonator received attention from the late Lord Rayleigh.

In the experimental investigation of the behaviour of these resonators, the microphone grid was placed in the short neck joining the inner and outer resonators. The grid formed one arm of a Wheatstone bridge, and the response of the resonator was measured by the deflection of a galvanometer. The deflection was, in all cases, very closely proportional to the average fall in resistance of the microphone grid. It is known† that this fall in resistance is proportional to the energy of the vibratory motion of the air surrounding the grid, provided that the amplitude of the motion is not too great.

§ 2. *Theory of the Boys Double Resonator.*—This type of double resonator consists essentially of two parts:—

(i) A resonator of the "stopped-pipe" variety, and
(ii) A resonator of the Helmholtz pattern.

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* 'Theory of Sound,' 2nd ed., vol. 2, p. 45.

† Tucker and Paris, *loc. cit.*, p. 410.

These are combined by inserting the neck of the Helmholtz resonator into a small hole in the stopped end of the pipe resonator. The arrangement is shown diagrammatically in fig. 1, where P is the pipe resonator and Q the Helmholtz resonator, communication between the two taking place through the short neck *c*, in which also is placed the hot-wire grid.

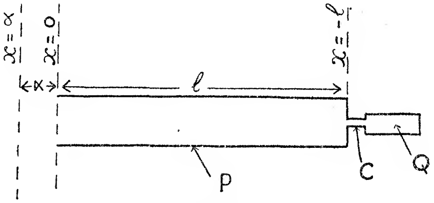

Fig. 1.

Such an arrangement has two resonant tones, which may be near together, or far apart, according to the relative dimensions of the various parts of the resonator. A formula will first be obtained which gives the frequencies of the resonant tones in terms of the cross-sectional area of the pipe, the hydrodynamical "conductance" of the communicating passage *c*, and the natural frequencies of the pipe and Helmholtz resonators. The method of investigation follows Rayleigh's treatment of the theory of resonance in pipes and other cavities.*

Let l be the length of the pipe resonator and α the "end correction," so that in the absence of the Helmholtz resonator the pipe would respond to wave-lengths given by

$$\tfrac{1}{4}(m\lambda) = l+\alpha = l' \; (m = 1, 3, 5, \text{etc.}).$$

If the pipe is circular in section α is approximately 0·8 of the radius. In what follows we shall confine our attention to the case when $m = 1$. Thus the fundamental note of the pipe is

$$n_1 = \lambda_1/a = 4l'/a \tag{1}$$

where a is the velocity of sound.

Let Q be the volume of the Helmholtz resonator and c the "conductance" of the neck, then the number of vibrations per second in the fundamental resonance pitch is given by Rayleigh's formula

$$n_2 = (a/2\pi) \sqrt{(c/Q)}. \tag{2}$$

Let the mouth of the pipe lie in the plane $x = 0$ (fig. 1), and the axis of the pipe be parallel to the axis of x. Then, if dissipation is left out of account, the velocity-potential inside the pipe may be written (as in the usual approximate theory of resonance in pipes),

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* 'Theory of Sound,' vol. 2, Chap. XVI.

$$\phi = \sin k(x-\alpha)\,\epsilon^{\iota\omega t}, \tag{3}$$

where $k = \omega/a$.

Near the stopped end of the pipe the motion must ultimately depart from the plane form represented by (3). Let $-\mathrm{L}$ be the greatest (negative) value of x for which the motion is still plane, and let $\phi_{-\mathrm{L}}$ be the value of ϕ when $x = -\mathrm{L}$. In the region, small compared with the wave-length, between the plane section of the pipe at $x = -\mathrm{L}$ and the interior of the Helmholtz resonator, the motion will be approximately that of an incompressible fluid. Thus, if dq/dt is the rate of transfer of fluid from the pipe into the Helmholtz resonator, we have, equating the flux through the pipe at $x = -\mathrm{L}$ to the flux in the neck,

$$\sigma\,(\partial\phi/\partial x)_{x=-\mathrm{L}} = dq/dt \tag{4}$$

where σ is the cross-sectional area of the pipe. Also, if ψ is the velocity-potential inside the Helmholtz resonator, we have

$$dq/dt = c\,(\phi_{-\mathrm{L}} - \psi) \tag{5}$$

and

$$\psi = \frac{a^2}{\mathrm{Q}}\int q\,dt \tag{6}$$

so that the equation of motion of the air in the neck is—

$$\frac{d^2q}{dt^2} + \omega_2{}^2 q = c\,\frac{d\phi_{-\mathrm{L}}}{dt}, \tag{7}$$

where $\omega_2 = 2\pi n_2$. The equation to determine the resonance frequencies is obtained by elimination of q between (4) and (7). Thus, since dq/dt and $\phi_{-\mathrm{L}}$ both vary as $\epsilon^{\iota\omega t}$, we obtain

$$\left(1 - \frac{\omega_2{}^2}{\omega^2}\right)\frac{dq}{dt} = c\phi_{-\mathrm{L}}. \tag{8}$$

From (3)

$$\phi_{-\mathrm{L}} = -\sin k\mathrm{L}' \,.\, \epsilon^{\iota\omega t}, \tag{9}$$

where $\mathrm{L}' = \mathrm{L} + \alpha$. Also

$$(\partial\phi/\partial x)_{x=-\mathrm{L}} = k\cos k\mathrm{L}' \,.\, \epsilon^{\iota\omega t}, \tag{10}$$

whence from (7)

$$dq/dt = \sigma k \cos k\mathrm{L}' \,.\, \epsilon^{\iota\omega t}. \tag{11}$$

By substitution from (9) and (11) in (8), the equation to determine the resonant tones is found to be

$$\sigma k\left(1 - \frac{\omega_2{}^2}{\omega^2}\right)\cos k\mathrm{L}' = -c\sin k\mathrm{L}',$$

or

$$\tan k\mathrm{L}' = -\frac{\sigma k}{c}\left(1 - \frac{\omega_2{}^2}{\omega^2}\right). \tag{12}$$

For comparison with experiment it is convenient to replace k, ω, and ω_2, by their equivalents in terms of n and n_2, the vibrations per second in the respective notes. Also the length L′ may be replaced by l', the "reduced length" of the pipe.

Since $k = \omega/a = 2\pi n/a$, and $l' = \frac{1}{4}\lambda_1 = a/4n_1$, equation (12) becomes

$$\tan\frac{\pi n}{2n_1} = -\frac{2\pi\sigma}{ac}n\left(1-\frac{n_2^2}{n^2}\right). \tag{13}$$

The vibrations per second in the resonant tones are given by the roots of (13), and are most conveniently found by a graphical method. For this purpose it is only necessary to plot the curves given by

$$y = \tan(\pi n/2n_1) \tag{14}$$

and

$$y = -\frac{2\pi\sigma}{ac}n\left(1-\frac{n_2^2}{n^2}\right). \tag{15}$$

An example is given in § 4.

§ 3. *Experimental Method.*—By means of the hot-wire microphone, equation (13) can easily be tested experimentally. In the experiment about to be described, the outer (pipe) resonator was made from two glazed earthenware pipes,* each of length 60 cm. and internal diameter 15 cm. These were cemented together to make one pipe 120 cm. long. A wooden "plunger" was fixed into this pipe at a distance of 104 cm. from one end. The plunger consisted of a wooden disc (P, fig. 2), 2·5 cm. thick and 14 cm. in diameter, from the middle of which was cut a circular hole 9 cm. in diameter. On the under surface of the wood was screwed a disc of brass (B) about 12 cm. in diameter. This brass disc carried at its centre a nipple (N) which was internally threaded, and into which the neck of the Helmholtz resonator was screwed, as shown in fig. 2. Between the brass plate and the wood was placed a washer of sheet rubber (R), which overlapped the wood and ensured a tight fit when the plunger was pushed into the pipe. The Helmholtz resonator consisted of a container (Q, fig. 2), made from brass tubing (internal diameter 5·4 cm.), into which was fitted a wooden piston (W), so that the volume, and therefore note, could be varied over a considerable range.

The microphone grid (M), and its holder, were of the same pattern as those used in previous experiments, and both are fully described in a paper to which reference has already been made.† The method of plotting resonance curves is also described in the same paper, and it is unnecessary to give more than a brief summary here. All observations were made out-of-doors, the resonator being placed at a distance of about 2 metres from the source of sound and supported on three wooden feet, so that the pipe stood vertically with the open end downwards and about 30 cm. above the ground. The microphone grid formed one arm of a Wheatstone bridge, and the deflections of the galvanometer (a microammeter with suitable series and shunt resistances) were used to measure the response of the resonator. The microphone grids carried a maximum heating current of 28 milliampères, and had a "hot" resistance of 350 ohms.

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* These pipes were used as a matter of convenience. A rectangular wooden box, open at one end, gives equally good results.

† Tucker and Paris, *loc. cit.*, pp. 391–393.

The source of sound was a siren of the Seebeck pattern, consisting of a brass siren disc mounted on the shaft of an electric motor. The disc was pierced with a ring of 15 holes, each 1·25 cm. in diameter, and spaced so that the distance from the centre of one hole to the centre of the next (measured along the circumference of the circle on which the centres of the holes lie), was 2·5 cm. The blast of air was supplied from a gas-compressor, the delivery pipe at the siren having the same diameter as the holes in the disc. It has been shown by Milne and Fowler* that when the holes are spaced in this manner (that is, when the distance from centre to centre is twice the diameter of a hole), the purity of the note given by the Seebeck siren is comparable with that of the "pure tone siren" designed by these authors. Some preliminary experiments showed that the theoretical conclusions are well borne out in practice, there being but little difference in quality between the tone of the Seebeck siren described above, and that of a 12-hole "pure tone siren" constructed as shown in Milne and Fowler's fig. 5.

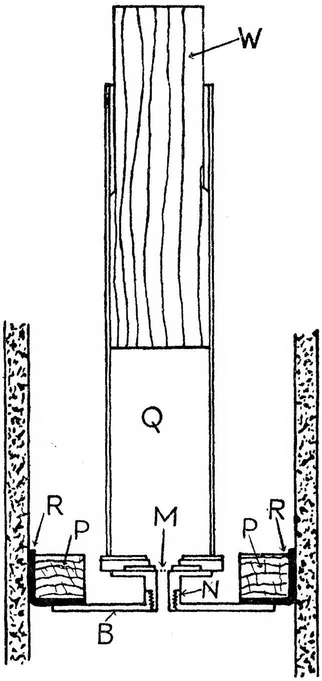

Fig. 2.

The speed of the motor, which could be regulated over a wide range by means of rheostats, was recorded by an Elliott speed-counter connected with the motor by flexible shafting.

§ 4. *Graphical Solution. Comparison of Calculated and Observed Frequencies.*—In order to calculate the resonant frequencies of the double resonator, it is necessary to know the values of n_1, n_2, σ and c. To find n_1, a pair of stethoscopes were connected by rubber tubing to the nipple N (fig. 2), and the speed of the siren necessary to give maximum resonance was determined by listening.

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* 'Roy. Soc. Proc.,' A, vol. 98, pp. 414–427 (1921).

It was found that n_1 was 78 vibrations per second at 19° C. From this the curve corresponding to equation (14), viz.,

$$y = \tan(\pi n/2n_1)$$

can be plotted, as in fig. 3.

The frequency n_2 of the Helmholtz resonator could be raised or lowered by moving the wooden piston W (fig. 2). In a preliminary experiment the frequency of the resonator corresponding to various positions of W was determined, and a scale of frequencies marked on it. The range of the scale was from 63 to 87 vibrations per second.

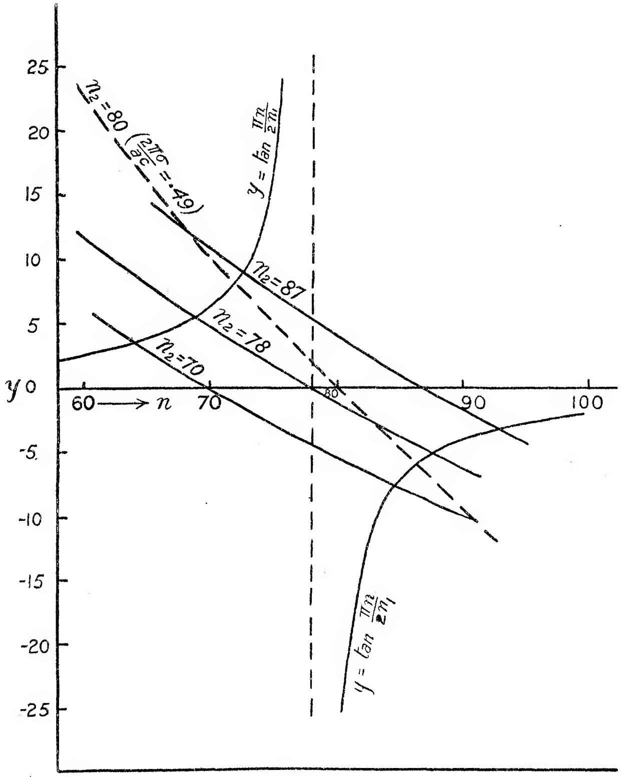

Fig. 3.

During this preliminary experiment the Helmholtz resonator was removed from the pipe and a brass flange similar to the plate B (fig. 2) screwed on to the neck, so that the conductance had the same value as when the resonator was in position.

Since the pipe was 15·2 cm. in diameter, we have

$$\sigma = 182 \text{ cms}^2.$$

To find c the microphone holder (with the flanged neck) was fitted to a container of known volume and the pitch determined experimentally. The value of c was then calculated from the formula (2) of § 2. The value found was 0·117 cm. From these values of σ and c, we get

$$2\pi\sigma/ac = 0{\cdot}286,$$

where a is the velocity of sound in air at 19° C. (the temperature at which the experiments were performed). So that (15) becomes

$$y = -0{\cdot}286n\left(1-\frac{n_2^2}{n^2}\right).$$

For comparison with the experimental results, the values 70, 78 and 87, were given to n_2 and the corresponding curves plotted as in fig. 3. The resonance frequencies of the double resonator are easily read off from the figure.

The experimental resonance curves (fig. 4) were obtained in the manner already described. Readings of the galvanometer deflection and the speed of the siren motor were taken as nearly as possible simultaneously. In fig. 4 the deflections are plotted against the siren note for six different values of n_2. When the maximum deflection occurred between two readings of the speed counter, it is indicated by a short horizontal line in the figure.

A comparison of figs. 3 and 4 shows that there is very good agreement

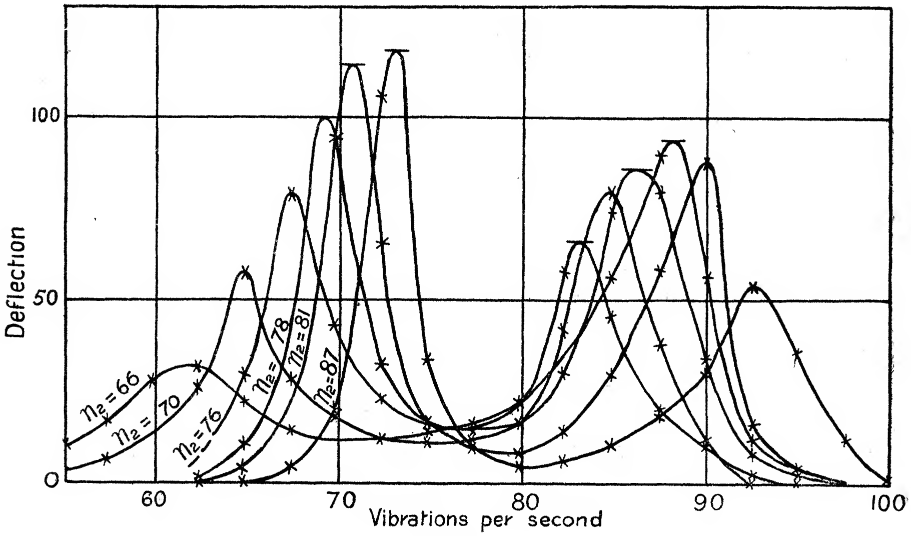

Fig. 4.

between the observed and calculated resonance frequencies. Some values of N_1 and N_2 taken from fig. 3 with the corresponding values as found by experiment, are given in the following Table:—

Table I.

n_2.	Calculated.		Observed.	
	N_1.	N_2.	N_1.	N_2.
70	84½	64	85	65
78	87½	69	88	69
87	93	73	93	73

§ 5. *The Tuning of the Component Resonators.*—The double resonator shows the usual characteristics of a coupled system, the components of which have the frequencies n_1 and n_2. Thus, if the pitch of the pipe is kept constant while the pitch of the Helmholtz resonator is slowly raised (by reducing the volume Q) from some value well below n_1, then, whatever the value of n_2, the two resonant tones are such that the lower one (N_2) is always below both n_1 and n_2, while the upper one (N_1) is always above them. There is one value of n_2 which make the difference N_1-N_2 a minimum. Let $2\pi\sigma/ac$ be fixed, so that the question is solely one of tuning the component resonators. If N_1 and N_2 were determined by the intersection of a straight line such as

$$y = \text{constant} \times (n-n_2)$$

with a tangent curve, the difference between N_1 and N_2 would be a minimum when $n_2 = n_1$. Now, in the region with which we are concerned, the curve corresponding to equation (15) does not depart very greatly from a straight line. In fact (15) may be written

$$y = -\frac{2\pi\sigma}{ac}\,\frac{n+n_2}{n}(n-n_2)$$

$$= -\frac{4\pi\sigma}{ac}(n-n_2) \qquad (16)$$

approximately, so that the condition for N_1 and N_2 to be nearest together is $n_2 = n_1$. This conclusion is confirmed by the result of the experiment already described. It was found that the minimum difference between N_1 and N_2 was 18¾ vibrations per second and occurred when n_2 was between 77 and 78. The minimum *interval* N_1/N_2, however, occurred when n_2 was above 78.

It may also be noted that the tilt of the curve (16) is given by the factor $4\pi\sigma/ac$, and for a given value of n_2 it is this factor that determines the coupling of the system and the relative positions of N_1 and N_2. If it is large

N_1 and N_2 are close together, while if it is small they are far apart. To illustrate this point, the results of a second experiment may be quoted, in which the pipe resonator had an internal diameter of 23 cm. and the conductance of the neck joining the two resonators was 0·153 cm. This gave, at the temperature of the experiment

$$2\pi\sigma/ac = 0{\cdot}491$$

instead of 0·286 as in the first experiment. The length of the pipe resonator was 100·3 cm., and it responded to a note of 79 vibrations per second. The effect of the increased ratio σ/c on the curve corresponding to (15) is shown by the broken line in fig. 3 which is worked out for the case when $n_2 = 80$. The intersection with the tangent curve gives

$N_1 = 86\frac{1}{2}$ vibrations per second.
$N_2 = 72$ „ „

while the observed values were

$N_1 = 86\frac{1}{2}$ vibrations per second.
$N_2 = 73\frac{1}{2}$ „ „

The calculated values should be rather low since the tangent curve is plotted for $n_1 = 78$ instead of 79, so that the close agreement between observed and calculated values is to some extent fortuitous.

Further observations showed that the minimum difference $N_1 - N_2$ was 13 vibrations per second and the minimum interval N_1/N_2 was 1·18. Both these minima occurred when n_2 was between 77 and 80, but their positions were not determined more exactly than this.

§ 6. *The Form of the Resonance Curves.*—Fig. 4 shows how the form of the resonance curves varies with the tuning of the component resonators. Let D_1 and D_2 represent the deflection of the galvanometer (approximately proportional to the energy of the vibration) when $n = N_1$, $n = N_2$ respectively. Both D_1 and D_2 varied in a well defined manner when (n_1 being constant) the pitch n_2 of the inner resonator is changed by altering its volume. When n_2 was either far above or far below n_1, D_1 and D_2 were relatively small, but passed through maximum values when n_2 was near to n_1. The growth of D_1 is clearly shown, a maximum occurring when $n_2 = 87$ The readings, however, do not extend far enough to show the passage of D_2 through a maximum, but it occurred approximately when n_2 was 87. It was also found that D_1 and D_2 were equal when n_2 was about 77, and that, as far as the experiment showed, D_1 was always greater or less than D_2 according as n_2 was less than or greater than 77.

In the second experiment, in which a pipe 23 cm. in diameter was used

($n_1 = 79$, $2\pi\sigma/ac = 0{\cdot}491$), the resonance curves had the same general character. The maximum value of D_1 occurred when n_2 was 73, and of D_2 when $n_2 = 89$, while D_1 and D_2 were equal when n_2 was 79.

The application of the foregoing results to the construction of a doubly-resonated microphone which will respond to a given range of frequencies presents no difficulty. The response will necessarily be unequal since—unless the damping is very large—the resonance curve will always show two maxima. In order to reduce the inequality to a minimum, the component resonators should be tuned so that $D_1 = D_2$. The minimum deflection occurring between $n = N_1$ and $n = N_2$ then depends only on the damping and on the ratio σ/c. For example, a microphone was constructed to respond to notes between 67 and 84 vibrations per second. The outer resonator was 23 cm. in diameter and responded to a note of 74 vibrations per second. The conductance of the neck joining the two resonators was 0·13 cm., so that $2\pi\sigma/ac$ was 0·59. When the inner resonator was tuned to 73 vibrations per second, the resonance curve showed two equal maxima at 70 and 82, while at no point between 67 and 84 was the deflection less than 40 per cent. of the maximum value.

§ 7. *Theory of the Helmholtz Double Resonator.*—This type of double resonator is shown diagrammatically in fig. 5. It consists of a Helmholtz resonator (c_1, Q_1) to which is attached a second Helmholtz resonator (c_2, Q_2) so that Q_1 and Q_2 communicate through c_2 while Q_1 communicates with the outside air through c_1. The whole resonator is supposed to be small compared with the wave-lengths of the notes to which it resounds. When a double resonator is used to obtain increased sensitivity to a particular tone, the Helmholtz double resonator is generally preferable to one of the Boys pattern.

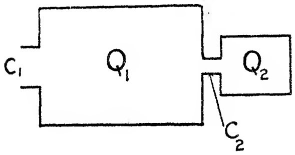

Fig. 5.

The theory of such resonators has been given by Rayleigh,* who obtained, by Lagrange's method, an expression from which the natural pitches can be calculated if c_1, c_2, Q_1, and Q_2 are known.† Alternatively, the same result may be obtained as follows:—

Let ϕ, ψ_1, and ψ_2 represent the velocity-potentials outside the resonator, inside Q_1 and inside Q_2 respectively. Then, if q_1 denotes the transfer of fluid through c_1 into Q_1, and q_2 denotes the transfer through c_2 from Q_1 to Q_2, we have

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* 'Theory of Sound,' vol. 2, p. 190; 'Phil. Trans.,' A, vol. 161, pp. 77–118 (1870); 'Phil. Mag.,' vol. 36, pp. 231–234 (1918); 'Scientific Papers,' vol. 1, Art. 5, and vol. 6, Art. 432.

† More exactly, it is only necessary to know the ratios c_1/Q_1, c_2/Q_2, and c_1/c_2 or Q_1/Q_2.

$$\frac{dq_1}{dt} = c_1(\phi-\psi_1), \qquad \frac{dq_2}{dt} = c_2(\psi_1-\psi_2), \tag{17}$$

$$\psi_1 = a^2\int\frac{q_1-q_2}{Q_1}\,dt, \qquad \psi_2 = a^2\int\frac{q_2}{Q_2}\,dt, \tag{18}$$

where a is the velocity of sound in the gas in the apertures c_1 and c_2. The equations of motion, obtained by elimination of ψ_1 and ψ_2 from (17) and (18) are

$$\left.\begin{aligned} \frac{d^2q_1}{dt^2}+\frac{a^2c_1}{Q_1}(q_1-q_2) &= c_1\frac{d\phi}{dt} \\ \frac{d^2q_2}{dt^2}+a^2c_2\left(\frac{q_2}{Q_2}-\frac{q_1-q_2}{Q_1}\right) &= 0 \end{aligned}\right\}. \tag{19}$$

To find the natural tones (without allowance for dissipation) let $d\phi/dt = 0$, and assume that q_1 and q_2 both vary as $\epsilon^{\iota\omega t}$. We thus obtain

$$\omega^4-a^2\omega^2\left(\frac{c_1+c_2}{Q_1}+\frac{c_2}{Q_2}\right)+a^4\frac{c_1c_2}{Q_1Q_2} = 0, \tag{20}$$

which is the equation given by Rayleigh.*

This result may be shown in a more convenient form.

Let $c_2 = \mu c_1$, $\omega = 2\pi n$, $n_1^2 = a^2c_1/4\pi^2Q_1$, and $n_2^2 = a^2c_2/4\pi^2Q_2$.

Then (20) may be written

$$n^4-n^2\{(\mu+1)n_1^2+n_2^2\}+n_1^2n_2^2 = 0. \tag{21}†$$

It is also convenient to write $n_\mu^2 = n_2^2+\mu n_1^2$, and it is easily seen that n_μ is the natural frequency of q_2 when $c_1 = 0$; that is, when all communication with the outer air is suppressed. If N_1^2 and N_2^2 are the roots of (21) regarded as a quadratic in n^2, we have

$$\left.\begin{matrix} N_1^2 \\ N_2^2 \end{matrix}\right\} = \tfrac{1}{2}(n_1^2+n_\mu^2)\pm\tfrac{1}{2}\{(n_1^2+n_\mu^2)^2-4n_1^2n_2^2\}^{\frac{1}{2}}. \tag{22}$$

Equation (22) allows us to calculate the frequencies of the resonant tones for given values of μ, n_1, and n_2. There is, however, a graphical method of finding N_1 and N_2 which is worth attention on account of the analogy with the case of the Boys resonator. Equation (21) may be written

$$\frac{n}{1-(n^2/n_1^2)} = -\frac{c_1}{c_2}n\left(1-\frac{n_2^2}{n^2}\right). \tag{23}$$

A comparison with equation (13) of § 2 shows that both (13) and (23) have the form

$$f(n, n_1) = -Kn(1-(n_2^2/n^2)).$$

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* 'Theory of Sound,' vol. 2, p. 191 (Equation (12)).

† Rayleigh, 'Sci. Papers,' vol. 6, p. 550.

For the Boys resonator,

$$\left.\begin{aligned} f(n, n_1) &= \tan(\pi n/2n_1) \\ \mathrm{K} &= 2\pi\sigma/ac \end{aligned}\right\}. \tag{24}$$

While (23) shows that for the double Helmholtz resonator

$$\left.\begin{aligned} f(n, n_1) &= \frac{n}{1-(n^2/n_1^2)} \\ \mathrm{K} &= c_1/c_2 \end{aligned}\right\}. \tag{25}$$

In both cases the value of y changes from positive to negative infinity as n passes through the value n_1. When n_1 and n_2 are given, the coupling and relative position of the resonant tones is determined by the parameter K. If K is large, N_1 and N_2 will be close together, whereas if it is small they will be far apart.

The actual values of N_1 and N_2 in the Helmholtz double resonator can be determined graphically by finding the intersections of the curves

$$y = \frac{n}{1-(n^2/n_1^2)} \tag{26}$$

and

$$y = -\frac{c_1}{c_2}\, n\,(1-(n_2^2/n^2)). \tag{27}$$

It may be observed that the equations of motion (19) are of the usual type for undamped force-coupled systems with two degrees of freedom. They may be written*

$$\left.\begin{aligned} \frac{d^2q_1}{dt^2}+\omega_1^2q_1+\gamma_1\omega_1^2q_2 &= c_1\frac{d\phi}{dt} \\ \frac{d^2q_2}{dt^2}+\omega_\mu^2q_2+\gamma_2\omega_\mu^2q_1 &= 0 \end{aligned}\right\}, \tag{28}$$

where $\gamma_1 = -1, \quad \gamma_2 = -\mu\omega_1^2/\omega_\mu^2 = -\mu n_1^2/n_\mu^2.$

The coefficient of coupling is $\sqrt{(\gamma_1\gamma_2)}$ or $(n_1/n_\mu)\sqrt{\mu}$. The analogous electrical circuit consists of two inductances (L_1 and L_2) and two capacities (C_1 and C_2)

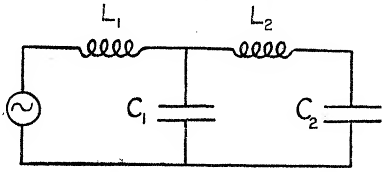

Fig.6.

coupled, as shown in fig. 6. In this analogy the electrical inductances correspond to the *reciprocals* of the hydrodynamical "conductances" in the acoustical circuit, while the electrical capacities correspond to the volumes of the acoustical resonators.* The coefficient of coupling in the electrical circuit is $\sqrt{\{C_2/(C_1+C_2)\}}$, whence, by analogy, the corresponding quantity in the acoustical circuit is $\sqrt{\{Q_2/(Q_1+Q_2)\}}$ or $(n_1/n_\mu)\sqrt{\mu}$.

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* *Cf.* M. Wien, 'Annalen der Physik,' vol. 61, p. 155 (1897).

§ 8. *Graphical Solution. Comparison with Experiments.*—The following experiment illustrates the formulæ of the preceding section. The general procedure was the same as that given in § 3. The outer resonator (c_1, Q_1) was made from a pipe 23 cm. in diameter (internal),† at one end of which was fixed a disc of wood, 2·5 cm. thick, and with a circular hole, 6·4 cm. in diameter, cut from the centre. A "plunger," similar to that shown in fig. 2 and carrying a Helmholtz resonator (c_2, Q_2), whose volume could be varied at

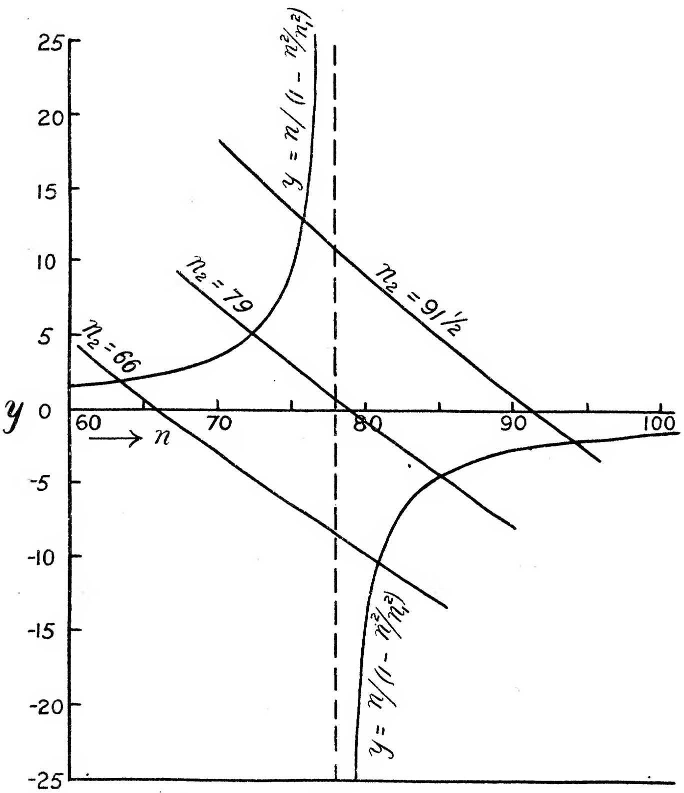

Fig. 7.

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* If the neck c_1 (fig. 5), instead of communicating with the outside air, is made to lead into a third enclosed volume, we obtain a coupled acoustical system which has been investigated by Hahnemann and Hecht, 'Physikalische Zeitschrift,' vol. 22, p. 357 (1921). An electrical analogue is described which is of a similar kind to that given above for the Helmholtz double resonator.

† A rectangular wooden box serves equally well.

will, was placed in the pipe at a distance of 49·5 cm. from the wooden disc. The microphone grid and its holder were of the same pattern as in the first experiment, and the pitch of the outer resonator was found to be 78 vibrations per second at 19° C. The curve corresponding to (26), with $n_1 = 78$, is plotted in fig. 7.

The conductances of the two orifices were—

$$c_1 = 4{\cdot}42 \text{ cms.}, \qquad c_2 = 0{\cdot}117 \text{ cm.},$$

so that $c_1/c_2 = 37{\cdot}8$.

The curves corresponding to (27) were worked out in three cases and are shown in fig. 7.

The experimentally obtained resonance curves are reproduced in fig. 8. In order to show the growth of the maxima at the resonance frequencies, the curves are given for six values of n_2, ranging from 60 to 91½. Some observed values of N_1 and N_2, corresponding to values taken from fig. 7, are shown in the following Table.

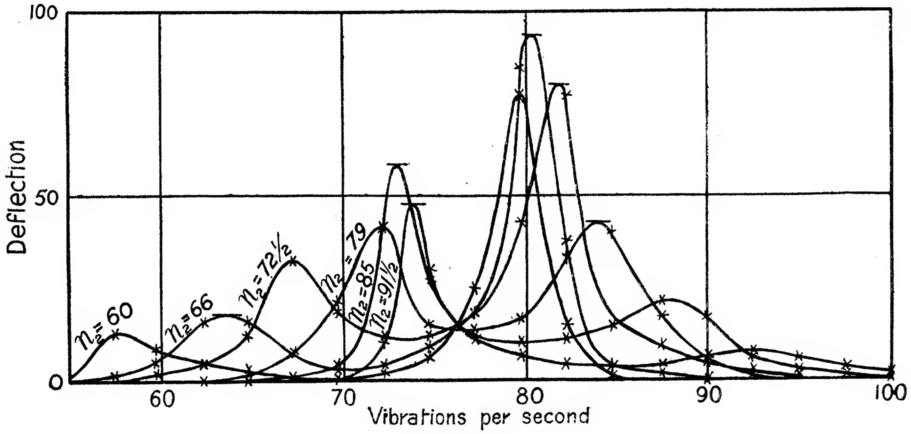

Fig. 8.

Table II.

n_2.	Calculated.		Observed.	
	N_1.	N_2.	N_1.	N_2.
66	81	63½	80½	63½
79	85	72	84	72
91½	93	75	93	74

§ 9. *Conditions for the Difference (or Interval) between the Resonant Tones to be a Minimum.*—The general behaviour of the Helmholtz double resonator when the tuning of the components is altered, is similar to that of the Boys resonator described in § 5. The question, however, as to how the component resonators should be tuned in order to make either $N_1 - N_2$ or N_1/N_2 a minimum, can be answered more definitely than in the case of the last-named resonator. Thus the condition for $N_1 - N_2$ to be a minimum when n_2 is constant and n_1 is varied, or *vice versa*, can be found from

$$(\partial/\partial n_1,\ \partial/\partial n_2)(N_1 - N_2) = 0.$$

On substituting the appropriate values of N_1 and N_2 from (22) it is found: (i) that when n_1 is constant $(N_1 - N_2)$ is a minimum when $n_2 = n_1$; and (ii) that when n_2 is constant $(N_1 - N_2)$ is a minimum when $n_1 = n_2/(1 + \mu)$.

By putting $n_2 = n_1$ in (22) it is found that the values of N_1 and N_2 when $(N_1 - N_2)$ is a minimum, n_1 being constant, are

$$\left.\begin{matrix} N_1 \\ N_2 \end{matrix}\right\} = \frac{n_1}{\sqrt{2}} \{\mu + 2 \pm \sqrt{(\mu[\mu+4])}\}^{\frac{1}{2}}. \tag{29}$$

The values of N_1 and N_2 in the corresponding case when n_2 is constant and n_1 is varied, can be obtained from (29) by writing $n_2/\sqrt{(1+\mu)}$ in place of n_1.

These results can be confirmed by reference to the experiment already described. In this experiment

$$n_1 = \text{constant} = 78,$$

$$\mu = 1/37{\cdot}8 = 0{\cdot}0265.$$

According to the above, $(N_1 - N_2)$ should therefore be a minimum when n_2 is 78, and the value of this minimum should be $84{\cdot}6 - 71{\cdot}9 = 12{\cdot}7$. The values found in the experiment were 78·5 and 12.

If μ is given, the interval N_1/N_2 is a function of the interval n_1/n_2 only. Let $p = n_2/n_1$. Then for N_1/N_2 to be a minimum we must have

$$(d/dp)(N_1/N_2) = 0,$$

from which, by means of (22), it is found that the required condition is

$$p = n_2/n_1 = \sqrt{(1+\mu)}. \tag{30}$$

From (30) and (22) it appears that the minimum value of N_1/N_2 is

$$\left[\frac{N_1}{N_2}\right] = \left\{\frac{\mu + 1 + \sqrt{(\mu[\mu+1])}}{\mu + 1 - \sqrt{(\mu[\mu+1])}}\right\}^{\frac{1}{2}}. \tag{31}$$

If $n_1 = 78$ and $\mu = 0{\cdot}0265$, as in the experiment described above, it is found from (30) and (31) that

$$n_2 = 79, \qquad [N_1 - N_2] = 1{\cdot}18.$$

The values found by experiment were 79 and 1·17.

§10. *Form of the Resonance Curves and Sensitivity.*—The form of the resonance curves depends on the coupling (measured by $(n_1/n_\mu)\sqrt{\mu}$), and damping of the component resonators. In the case of the singly resonated microphone, the dissipation of energy can be represented in the equation of motion by a term proportional to the instantaneous current of air in the neck of the resonator.* Similarly, the dissipation of energy in the double resonator can be represented by terms proportional to the currents in the apertures c_1 and c_2. When these terms are introduced, the equations of motion become

$$\left.\begin{aligned} \frac{d^2q_1}{dt^2}+2h_1\frac{dq_1}{dt}+\omega_1^2q_1-\omega_1^2q_2 &= \frac{c_1\mathrm{F}}{\rho}\epsilon^{\iota\omega t} \\ \frac{d^2q_2}{dt^2}+2h_2\frac{dq_2}{dt}+\omega_\mu^2q_2-\mu\omega_1^2q_1 &= 0 \end{aligned}\right\}, \qquad (32)$$

where F is the pressure amplitude due to the source (supposing the mouth of the resonator to be closed), ρ is the density of the air, and h_1 and h_2 are due to the operation of viscous forces in the apertures c_1 and c_2, and to the escape of acoustical energy into the surrounding air. The proportions in which these causes contribute to h_1 and h_2 depend on the exact size and shape of the apertures concerned, but whatever they may be, h_1 and h_2 will be functions of the frequency of the vibrations.

The equations (32) have the usual well-known solutions.† In the present case we are concerned only with the energy of the forced vibration in the second aperture, which is proportional to $\left|\frac{dq_2}{dt}\right|^2$, and it is easily shown from (32) that

$$\left|\frac{dq_2}{dt}\right|^2 = \frac{\omega^2\omega_1^4c_2^2\mathrm{F}^2/\rho}{[\{(\omega_1^2-\omega^2)(\omega_\mu^2-\omega^2)-\mu\omega_1^4-4h_1h_2\omega^2\}^2+4\omega^2\{h_1(\omega_\mu^2-\omega^2)+h_2(\omega_1^2-\omega^2)\}^2]}. \qquad (33)$$

Approximate numerical values for h_1 and h_2 can be found by assuming that they are constant for the range of frequencies contained in the resonance curve. Thus h_2 can be found by plotting the resonance curve of the inner resonator by itself, and when h_2 is known, an approximate value for h_1 can be obtained from (33), by finding what value will produce the best agreement between the observed and calculated resonance curves. For example, in the experiment described in §8, it was found in this way that approximate values were $h_1 = 4$ and $h_2 = 20$.

It has previously been pointed out,‡ that in narrow apertures (such as c_2 in the foregoing experiment) the dissipation due to the diffusion of acoustical energy into the surrounding air contributes only a negligibly small amount to the damping factor. The case of a wide aperture, such as c_1, however, is somewhat different. This aperture had a diameter of 6·4 cm., and a conductance of 4·4 cm. The damping factor, if viscosity were left out of account, would be $\omega^2 c/2\pi a$ (supposing the opening to be flanged), or 2·3 if $\omega/2\pi = 75$ vibrations per second. Thus the escape of energy from the resonator accounts for more than half of the experimentally determined value of h_1.

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* Tucker and Paris, *loc. cit.*, pp. 398–399.

† See M. Wien, *loc. cit.*, p. 181.

‡ Tucker and Paris, *loc. cit.*, p. 399.

By making use of these approximate values of h_1 and h_2, it can be shown from (33) that a doubly-resonated microphone of the type described may be more than one hundred times as sensitive (to one of its resonant tones) as a suitably tuned singly-resonated microphone with an aperture similar to that of the inner resonator. This remarkable increase in sensitivity is in accordance with experience.

It was pointed out by Rayleigh* that, for great sensitivity, the inner resonator must be small compared with the outer one. In Rayleigh's investigation (which takes no account of dissipation) it is shown that if n_1 and n_2 are equal, and if Q_2 is small compared with Q_1, then the ratio of the maximum velocity in the second aperture to that in the first is given by

$$V_2/V_1 = (Q_1/Q_2)^{3/2}. \qquad (34)$$

According to this result, V_2 could be made to exceed V_1 to any extent simply by making Q_2 small enough. It is more satisfactory, however, to make use of (33), by means of which the values of V_2 can be compared under different circumstances without reference to V_1. Moreover, it is found in practice that the effect of dissipation is by no means negligible, for when Q_2 is diminished it is necessary also to reduce c_2 (the conductance of the second aperture) in order that the ratio c_2/Q_2, and hence n_2, may be maintained at its former value. This can be done either by increasing the length of the opening or by reducing its cross-sectional area. In either case, the damping factor h_2 will be increased, and this, as shown by (33), tends to neutralise the advantage gained by the reduction in Q_2. The effect of viscosity becomes important when the second aperture is made very small, and for this reason there is probably no advantage in making the cross-section (supposed circular) much less than 3 or 4 mm. in diameter.

In hot-wire microphones, the extent to which the cross-section of the second aperture can be advantageously reduced is limited by the size of the grid. The smallest openings that have been used by the author were 3 mm. in diameter (internal) and between 2 and 3 mm. long. These were used in double resonators constructed to respond to notes of 250 vibrations per second and upwards. Besides being very sensitive, these microphones had the additional advantage that the hot-wire grid was well screened from draughts and wind.

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* 'Phil. Mag.,' vol. 36, pp. 231–234 (1918); 'Sci. Papers,' vol. 6, pp. 549–551.

In the case of larger apertures, the increase in sensitivity obtained by a reduction of the cross-sectional area is very marked. Thus, in a certain experiment, the second aperture was merely the circular hole in the mica plate which is used to carry the grid, while the "grid" itself was a single strand of wire fastened across the opening. The inner and outer resonators were each tuned to 108 vibrations per second. In the first part of the experiment, the diameter of the aperture was 1 cm., and this was reduced in the second part to 0·5 cm. The effect of this reduction was to increase the response (measured in ohms change per centimetre of wire exposed) to five times its first value at the higher note, and three times at the lower note. Precautions were taken to ensure that the amplitude of the sound affecting the microphone was the same in each case, and that the enhanced effect was due to an increase in the acoustical, as distinct from the electrical sensitivity.

§ 11. *Summary.*—The paper records the result of investigations into the properties of double resonators which can be used with hot-wire microphones in order to increase sensitivity, and also, if desired, to widen the range of response. Two types of resonator are dealt with, namely:—

(i) The "Boys double resonator," consisting of a "stopped pipe" in series with a Helmholtz resonator. The theory is given and experiments are described which illustrate the results obtained.

(ii) The "Helmholtz double resonator," consisting of two Helmholtz resonators in series. Rayleigh's theory is extended and experiments are described, the results of which show a satisfactory agreement with the theoretical formulæ. The form of the resonance curves is discussed, and the conditions affecting the sensitivity.

§ 12. *Conclusion.*—The greater part of the work described in this paper was done at the Signals Experimental Establishment, Woolwich, and is now published by permission of the War Office.

The author is indebted to Major W. S. Tucker for much useful advice and assistance.